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Partial Knowledge Restrictions on the Two-Stage Threshold Model of Choice

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Abstract

In the context of the two-stage threshold model of decision making, with the agent's choices determined by the interaction of three “structural variables,” we study the restrictions on behavior that arise when one or more variables are exogenously known. Our results supply necessary and sufficient conditions for consistency with the model for all possible states of partial knowledge, and for both single- and multi-valued choice functions.

1 Introduction

Recent work in the theory of individual choice behavior has modified the classical preference maximization hypothesis in various ways. One approach has been to weaken the consistency properties that preferences are ordinarily assumed to possess.¹ Another has been to study relationships between preference and choice other than straightforward maximization.² And a third has been to permit additional, non-preference-related factors—as well as multiple preferences—to influence decision making in some way.³

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¹For example, Eliaz and Ok [9], Mandler [14], Nishimura and Ok [23], and others allow preferences to be incomplete, following in the tradition of Aumann [2] and Bewley [5].

²Models of this sort have been axiomatized by Baigent and Gaertner [3], Eliaz et al. [10], Mariotti [19], and Tyson [30], among others.

³In addition to the contributions cited below, we have for example the work of Bossert and Sprumont [6] and Masatlioglu and Ok [22] on status-quo bias; Ambrus and Rozen [1] and Rubinstein and Salant [25] on multi-self and framing models; Caplin and Dean [7],

In the context of this literature, the revealed preference exercise required to characterize a given model can be quite challenging, since multiple factors must often be inferred simultaneously from behavior. Moreover, models with more than one component make possible a variant of the usual characterization problem: An outside observer can test a collection of choice data for consistency with the model *while treating one or more components as known*.

For example, suppose that we postulate a decision maker who maximizes a utility function over the alternatives that he or she notices, but pays attention only to those options with a sufficiently high level of salience (with regard to the visual or another sensory system). If salience is directly measurable, then the relevant question is whether these measurements and the choice data together can be reconciled with our behavioral hypothesis.⁴ And this means, of course, finding suitable assignments of the unobserved components—namely, the utility function and the salience thresholds.

As another example, imagine a choice among lotteries by a satisficing agent who decides between the options deemed satisfactory by following a social-norm ordering. On the one hand, the social norm might be known to the theorist, in which case it and the choice data must be jointly reconciled with the model by specifying the utilities and satisficing thresholds. Alternatively, perhaps the norm is unknown but we wish to introduce a maintained assumption of risk neutrality. In the latter case our search will be for satisficing thresholds (relative to expected value) plus a social norm that together generate the observed behavior.

Evidently, questions of this sort can be posed for any multiple-component model of choice, with any subset of the components taken to be known. In an electoral setting we might plausibly know the economic interests of a voter but not his or her ideology, while in a managerial setting we might assume profit maximization subject to an unobserved market-share constraint. Note that a model component could be designated as “known” due to an assumption, a physical observation, econometric estimates from a separate data set, or background knowledge of the agent’s environment, among other reasons.

In this paper we explore the issue of testing model consistency under partial knowledge—one that appears to be largely unexamined within axiomatic choice theory. To give this enterprise some concreteness, we shall commit to a particular model of how choices are determined by the interaction of various factors. We adopt a framework that is deliberately very general, and can

Cherepanov et al. [8], and Masatlioglu and Nakajima [20] on search and consideration sets; and Mandler et al. [15], Manzini and Mariotti [17], and Bajraj and Ülkü [4] on procedural models.

⁴The observer might be able to determine salience levels, say, using knowledge of the physiology of vision and the spatial arrangement of the choice alternatives.

accommodate each of the above examples. For a given menu A of options, the “two-stage threshold” (TST) model of choice specifies that the agent will select an alternative that solves

$$\max_{x \in A} g(x) \text{ subject to } f(x) \geq \theta(A). \quad (1)$$

Here the model components, which we shall call “structural variables,” are real-valued functions f and g defined on the space of alternatives, plus a real-valued function θ defined on the space of menus.

The TST framework has no fixed interpretation. Indeed, the model overlaps with several existing theories based on very different hypotheses about the process of decision making. One possibility is to interpret f as a measure of consideration or attention priority, θ as a cognition-threshold map, and g as a utility function; as in the contributions of Lleras et al. [13] and Masatlioglu et al. [21].⁵ Another possibility is to interpret f as the utility function, θ as a utility-threshold map, and g as a salience measure; as in Tyson [31]. Under these two interpretations the first stage of the model captures, respectively, the “consideration set” (a concept from the marketing literature) and Simon’s [27] notion of satisficing.⁶

In its general form the TST model has been characterized by Manzini et al. [18], who demonstrate that Equation 1 can accommodate a wide range of behavior patterns. Indeed, when each set of acceptable choices is required to be a singleton, it is straightforward to show that *any* observed data set can be generated by the model (see Proposition 2.6). Moreover, even if we allow multiple acceptable choices, the constraints imposed by the framework itself remain conspicuously weak (see Theorem 2.5). While the theories mentioned above reduce this freedom by imposing specialized restrictions on the structural variables, our approach at present is to fix one or more variables completely and leave the others entirely unconstrained.⁷ We then seek to identify the forms of behavior that remain consistent with the model.

Given a particular interpretation of the model, some structural variables will be more naturally assumed to be known than others. Since our intention is to avoid favoring any specific viewpoint, we provide a complete and hence interpretation-free collection of characterization results: For any strict subset of the three structural variables, we supply necessary and sufficient conditions for behavior to be compatible with the TST model when the variables in the

⁵Related models are studied by Eliaz and Spiegler [11] and Spears [28].

⁶For further details of these interpretations of the TST framework, see [18, pp. 879–881].

⁷These two approaches can also be combined. For instance, Theorem 3.12 below can be modified to incorporate the “expansiveness” restriction on $\langle f, \theta \rangle$ imposed by Tyson [31].

subset are known and all others are unrestricted.⁸ This collection of results—together with posing the partial knowledge question for multiple-component choice models—makes up the contribution of the paper.

Broadly speaking, our analytical method is to use the choice data together with the known variables to infer as much information as we can about the unobserved variables. We then look for ways in which this information could be self-contradictory, and formulate axioms that rule them out. Such axioms will always be necessary for behavior to be compatible with the model. And if our search for contradictions is thorough enough, they will also be sufficient (though demonstrating this may require extended arguments).

For example, suppose that g is known while both f and θ are unobserved (cf. Theorem 3.12). If alternatives x and y are both on menu A , and if also $g(x) > g(y)$, then clearly x and y cannot both be chosen from A . This is the simplest illustration of how choice data and a known structural variable together can lead to a contradiction, which must be ruled out axiomatically.

Suppose now that f and g are both known, with only θ unobserved (cf. Theorem 3.18). Since g is known, the variety of contradiction seen in the preceding paragraph must still be avoided. Furthermore, if alternatives x and y are both on menu A , and if also $f(x) \geq f(y)$ and $g(x) \geq g(y)$, then we cannot have that y is chosen from A unless x too is chosen. These two types of contradictions turn out to exhaust the implications of the model when both f and g are known, which is to say that axioms ruling them out provide the desired characterization.

The remainder of the paper is structured as follows. Section 2 defines the TST framework and reviews the axiomatization of the unconstrained model given by Manzini et al. [18]. Our novel results are stated first in Section 3 for multi-valued choice functions, and then in Section 4 for the single-valued special case. Section 5 contains a brief concluding discussion. Proofs of the general (multi-valued) versions of our results can be found in the Appendix.

⁸We assume that knowledge of one structural variable has no direct implications for the unknown variables, which can be chosen arbitrarily to generate the observed behavior. This assumption will not hold under interpretations of the model that motivate joint restrictions on the variables. For example, in [31] the functions f and θ are linked by the property of “expansiveness.” It is even possible that knowledge of one variable could completely determine another, for instance if $\theta(A)$ equals the average $|A|^{-1} \sum_{x \in A} f(x)$ of the available f -values. Dependencies like these could certainly be taken into account in the characterization exercises we carry out, but we shall not impose any such link between structural variables as a blanket restriction on the model.

2 The two-stage threshold model

Let X be a nonempty, finite set, and let $\mathcal{D} \subseteq \mathcal{A} = 2^X \setminus \{\emptyset\}$. The elements of X are called *alternatives*, the elements of \mathcal{D} are called *menus*, and any map $C : \mathcal{D} \rightarrow \mathcal{A}$ such that $\forall A \in \mathcal{D}$ we have $C(A) \subseteq A$ is called a *choice function*. The *choice set* $C(A)$ contains the alternatives that are chosen from menu A . A choice function is *single-valued* if it returns only singleton choice sets. Without loss of generality, we shall assume that $\forall x \in X$ we have $\{x\} \in \mathcal{D}$.

In the TST model, the choice set associated with menu A is constructed by maximizing $g(x)$ subject to $f(x) \geq \theta(A)$. Here $f : X \rightarrow \mathbb{R}$ is the *primary criterion*, $g : X \rightarrow \mathbb{R}$ the *secondary criterion*, and $\theta : \mathcal{D} \rightarrow \mathbb{R}$ the *threshold map*. These three components of the model are termed *structural variables*, any triple $\langle f, \theta, g \rangle$ is a *profile*, and any pair $\langle f, \theta \rangle$ is a *primary profile*.

Given a primary profile $\langle f, \theta \rangle$ and an $A \in \mathcal{D}$, write $\Gamma(A|f, \theta) = \{x \in A : f(x) \geq \theta(A)\}$ for the subset of available alternatives whose primary criterion values are above the relevant threshold. The TST model can now be defined formally as follows.

2.1 Definition. A *two-stage threshold representation* of C is a profile $\langle f, \theta, g \rangle$ such that $\forall A \in \mathcal{D}$ we have $C(A) = \operatorname{argmax}_{x \in \Gamma(A|f, \theta)} g(x)$.

In order to axiomatize this model, Manzini et al. [18] use several binary relations that are revealed by the agent's choices. The *separation relation* encodes situations where one alternative is chosen and a second (available) alternative is rejected.

2.2 Definition. Let xSy if $\exists A \in \mathcal{D}$ such that $x \in C(A)$ and $y \in A \setminus C(A)$.

The *togetherness relation* encodes situations where two alternatives both are chosen, and its transitive closure is the *extended togetherness relation*.⁹

2.3 Definition. Let xTy if $\exists A \in \mathcal{D}$ such that $x, y \in C(A)$, and let xEy if $\exists z_1, z_2, \dots, z_n \in X$ such that $x = z_1 T z_2 T \dots T z_n = y$.

Finally, the *first-stage separation relation* encodes separations that must be attributed to the primary criterion, since an extended togetherness relationship guarantees equal values of the secondary criterion.¹⁰

2.4 Definition. Let xFy if xEy and xSy .

⁹Recall that a relation R is transitive if $xRyRz \Rightarrow xRz$, and that the transitive closure of R is the smallest transitive relation containing it.

¹⁰More explicitly, if xEy then $x = z_1 T z_2 T \dots T z_n = y$, which under the TST model implies that $g(x) = g(z_1) = g(z_2) = \dots = g(z_n) = g(y)$ for some $z_1, z_2, \dots, z_n \in X$. If also xSy , then under the model we must have $f(x) \geq \theta(A) > f(y)$ for some $A \in \mathcal{D}$.

Manzini et al. [18, pp. 876–879] prove that acyclicity of this last relation is the one and only condition needed to characterize the TST model in the absence of known structural variables.¹¹

2.5 Theorem. *A choice function has a two-stage threshold representation if and only if the relation F is acyclic.*

Moreover, since the relations T , E , and F are all empty in the single-valued case, here the acyclicity condition holds trivially and so consistency with the model is assured.

2.6 Proposition. *Any single-valued choice function has a two-stage threshold representation.*

3 Characterization results

3.1 One known structural variable

We consider first situations where exactly one of the three structural variables is known. As explained in Section 1, our method is to search for information about the unknown variables that is revealed by the choice data together with the known variable. We then construct axioms that prevent contradictions, which will yield the desired characterization once all possible contradictions have been identified and ruled out.

Known primary criterion. Suppose that the primary criterion f is known, while the threshold map θ and secondary criterion g are not, and let $x, y \in A$. In this case knowledge that $f(y) \geq f(x)$ implies that y survives the first stage of choice from A if x does so; i.e., that $x \in \Gamma(A|f, \theta) \Rightarrow y \in \Gamma(A|f, \theta)$. But then from the observation xSy we can deduce that $g(x) > g(y)$, since x can only have been separated from y at the second stage. In other words, second-stage superiority of one alternative over another is revealed by the relation defined as follows in terms of the known f and the observed C .

3.1 Definition. Let xH_fy if $f(y) \geq f(x)$ and xSy .

Since H_f implies *strict* second-stage superiority between alternatives, this relation must be acyclic for the model to hold. That is to say, acyclicity of H_f is necessary for C to admit a TST representation consistent with the partial profile $\langle f, \cdot, \cdot \rangle$.

¹¹Recall that a relation R is acyclic if $x_1Rx_2R\cdots Rx_n \Rightarrow x_1 \neq x_n$.

3.2 Example. Let $f(x) = f(y) = 1$, $C(xy) = x$, and $C(xyz) = y$. If $\langle f, \theta, g \rangle$ were a TST representation of C , then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, but at the same time $f(x) \geq f(y)$ and $C(xyz) = y$ would imply $g(y) > g(x)$, a contradiction.

While H_f reveals second-stage superiority, the extended togetherness relation E reveals second-stage indifference (see Footnote 10). Choice data can thus be incompatible with the TST model even in the absence of an H_f -cycle, as seen in the following example.

3.3 Example. Let $f(x) = 1$, $f(y) = 2$, $f(z) = 0$, $C(xy) = x$, $C(xz) = z$, and $C(yz) = yz$. If $\langle f, \theta, g \rangle$ were a TST representation of C , then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, and likewise $f(x) \geq f(z)$ and $C(xz) = z$ would imply $g(z) > g(x)$. But then $C(yz) = yz$ would imply $g(y) = g(z) > g(x) > g(y)$, a contradiction.

The choice function in the latter example has the relation F empty (and therefore vacuously acyclic), so there is no difficulty in exhibiting a TST representation when all three structural variables are free. It is only in combination with the specified f that the data in C conflict with the model, due to the mixed cycle zH_fxH_fyEz .

With such situations in mind, we define formally the relationship of being linked by a chain of alternatives connected sequentially by either H_f or E .

3.4 Definition. Let xW_fy if $\exists z_1, z_2, \dots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \dots, n-1\}$ we have either $z_kH_fz_{k+1}$ or z_kEz_{k+1} .

This relation reveals weak second-stage superiority, and strict superiority if at least one link in the chain is via H_f . The condition needed for a characterization, analogous to Richter's [24, p. 637] Congruence axiom, is then that no alternative bear W_f to itself except via extended togetherness.

3.5 Theorem. A choice function has a two-stage threshold representation consistent with $\langle f, \cdot, \cdot \rangle$ if and only if $xW_fy \Rightarrow \neg[yH_fx]$.

Known threshold map. Now suppose instead that the threshold map θ is the known structural variable. In this situation, if we can identify $A, B \in \mathcal{D}$ and $y \in A \cap B$ such that both $y \in C(B)$ and $\theta(B) \geq \theta(A)$, then clearly we can conclude that $f(y) \geq \theta(A)$. If moreover both $x \in C(A)$ and $y \notin C(A)$, then y must have been eliminated from A at the second stage and hence we must have $g(x) > g(y)$.

To capture this method of deducing second-stage superiority from the known θ and observed C , we define the *critical threshold* for alternative y .

3.6 Definition. Let $M(y|\theta) = \max\{\theta(A) : A \in \mathcal{D} \wedge y \in C(A)\}$.

In other words, the critical threshold is the highest threshold of any menu to whose choice set the alternative belongs, with the obvious consequence that $y \in C(A)$ only if $M(y|\theta) \geq \theta(A)$.¹² The above logic is then expressed in the construction of the following revealed relation.

3.7 Definition. Let $xH_\theta y$ if $\exists A \in \mathcal{D}$ such that $M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$.

As before, we can use the strict second-stage relation H_θ to define a weak counterpart.

3.8 Definition. Let $xW_\theta y$ if $\exists z_1, z_2, \dots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \dots, n-1\}$ we have either $z_k H_\theta z_{k+1}$ or $z_k E z_{k+1}$.

And our characterization result then once again uses a congruence condition to identify the data sets consistent with the model.

3.9 Theorem. *A choice function has a two-stage threshold representation consistent with $\langle \cdot, \theta, \cdot \rangle$ if and only if $xW_\theta y \Rightarrow \neg[yH_\theta x]$.*

Known secondary criterion. If the known structural variable is the secondary criterion g , then the following relation—defined analogously with H_f above—will reveal strict first-stage superiority.

3.10 Definition. Let $xH_g y$ if both $g(y) \geq g(x)$ and xSy .

This new relation must clearly be acyclic. But since it pertains to the first stage, combining it with extended togetherness (as in the construction of W_f and W_θ) is unhelpful. Instead, we need to check that *revealed* second-stage indifference agrees with the *observed* secondary criterion, which is to say that alternatives related by E have identical g -values.

3.11 Example. Let $g(x) = 1$, $g(y) = 2$, $C(xz) = xz$, and $C(xyz) = yz$. If $\langle f, \theta, g \rangle$ were a TST representation of C , then $C(xz) = xz$ would imply that $g(x) = g(z)$, while $C(xyz) = yz$ would imply that $g(z) = g(y)$. But then we would have $g(x) = g(y)$, which is false.

3.12 Theorem. *A choice function has a two-stage threshold representation consistent with $\langle \cdot, \cdot, g \rangle$ if and only if H_g is acyclic and $xEy \Rightarrow g(x) = g(y)$.*

¹²Note that since for each $y \in X$ we have $\{y\} \in \mathcal{D}$ and $y \in C(\{y\})$, and since \mathcal{D} is a finite set, the critical threshold is always well defined.

3.2 Two known structural variables

Next we consider situations where exactly two structural variables are known.

Known primary profile. When the full primary profile $\langle f, \theta \rangle$ is known, it is possible for this information to contradict the choice function directly, in a way that has nothing to do with inferences about the second stage.

3.13 Example. Let $f(x) = 1$, $f(y) = 3$, $\theta(xyz) = 2$, and $C(xyz) = x$. If $\langle f, \theta, g \rangle$ were a TST representation of C , then $x \in C(xyz)$ would imply that $f(x) \geq \theta(xyz)$, which is false.

To avoid such contradictions, our first necessary condition says simply that for each $x \in C(A)$ we have $f(x) \geq \theta(A)$; abbreviated as $f[C(A)] \geq \theta(A)$.

With the primary criterion known, the statement $M(y|\theta) \geq \theta(A)$ in the definition of H_θ (guaranteeing that y will survive the first stage of choice from menu A) can be replaced with an explicit assumption that $f(y) \geq \theta(A)$. This modification leads to the following relation, which continues to reveal strict second-stage superiority.

3.14 Definition. Let $xH_{f\theta}y$ if $\exists A \in \mathcal{D}$ such that $f(y) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$.

The weak counterpart to this relation is defined in the usual way.

3.15 Definition. Let $xW_{f\theta}y$ if $\exists z_1, z_2, \dots, z_n \in X$ such that $z_1 = x$, $z_n = y$, and for each $k \in \{1, 2, \dots, n-1\}$ we have either $z_k H_{f\theta} z_{k+1}$ or $z_k E z_{k+1}$.

And the resulting congruence axiom then completes our characterization for the present case.¹³

3.16 Theorem. A choice function has a two-stage threshold representation consistent with $\langle f, \theta, \cdot \rangle$ if and only if $xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x]$ and $f[C(A)] \geq \theta(A)$.

Known primary and secondary criteria. Now suppose that both f and g are known, but θ is unobserved. Here $xH_f y$ continues to imply that $g(x) > g(y)$. But rather than merely being checked for cycles and contradictions with the choice function, as in Theorem 3.5, inequalities of this sort can now be tested directly against the known g .

¹³Unlike Theorems 3.5 and 3.9, Theorem 3.16 can be viewed as a direct consequence of Richter's [24] classical axiomatization. This is because when the entire primary profile is observable, the subsets $\Gamma(A|f, \theta)$ of alternatives that survive the first stage are themselves observable. Provided $f[C(A)] \geq \theta(A)$, these survivor subsets can be treated as surrogate menus, and the TST characterization problem reduces to the classical revealed-preference exercise considered by Richter.

3.17 Example. Let $f(x) = 1$, $f(y) = 2$, $g(x) = g(y) = 0$, and $C(xy) = x$. If $\langle f, \theta, g \rangle$ were a TST representation of C , then $f(y) \geq f(x)$ and $C(xy) = x$ together would imply $g(x) > g(y)$, which is false.

Moreover, since the secondary criterion is known, the condition that options related by E have identical g -values (used in Theorem 3.12) remains necessary for a TST representation. Together, these two consistency tests supply the desired characterization.

3.18 Theorem. *A choice function has a two-stage threshold representation consistent with $\langle f, \cdot, g \rangle$ if and only if $xH_f y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$.*

Known threshold map and secondary criterion. If instead θ and g are known, but not f , then we must substitute H_θ for H_f as our indicator of second-stage superiority. With this modification, our search for contradictions proceeds just as in the previous case, yielding the following result.

3.19 Theorem. *A choice function has a two-stage threshold representation consistent with $\langle \cdot, \theta, g \rangle$ if and only if $xH_\theta y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$.*

4 Single-valued choice functions

In this section we specialize our results to the context of single-valued choice functions. This restriction simplifies our characterizations, albeit at the cost of substantial generality.

To understand the impact of the single-valuedness restriction, it is useful to partition our results into two groups: those where the secondary criterion is known, and those where it is unknown. Within the first group, each of the three axiomatizations (namely, Theorems 3.12, 3.18, and 3.19) involves the condition that $xEy \Rightarrow g(x) = g(y)$. But since the relation E is always empty for single-valued C , this condition now holds vacuously and can be deleted. Our simplified characterizations then appear as follows.

4.1 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle \cdot, \cdot, g \rangle$ if and only if H_g is acyclic.*

4.2 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle f, \cdot, g \rangle$ if and only if $xH_f y \Rightarrow g(x) > g(y)$.*

4.3 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle \cdot, \theta, g \rangle$ if and only if $xH_\theta y \Rightarrow g(x) > g(y)$.*

Now consider the second group of results, with g unknown. Each of these axiomatizations (namely, Theorems 3.5, 3.9, and 3.16) involves a congruence condition stating that an alternative can be revealed weakly second-stage superior to itself only via the extended togetherness relation. But again, the relation E is always empty in the single-valued case, so our congruence conditions each reduce to acyclicity of the relevant notion of strict second-stage superiority. We thus obtain the following simplified characterizations.

4.4 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle f, \cdot, \cdot \rangle$ if and only if H_f is acyclic.*

4.5 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle \cdot, \theta, \cdot \rangle$ if and only if H_θ is acyclic.*

4.6 Proposition. *A single-valued choice function has a two-stage threshold representation consistent with $\langle f, \theta, \cdot \rangle$ if and only if $H_{f\theta}$ is acyclic and $f[C(A)] \geq \theta(A)$.*

Under the restriction to single-valued choice functions, we find that the TST model with one known structural variable is characterized in each case by a simple acyclicity condition (Propositions 4.1, 4.4, and 4.5). If either of the first-stage variables is known, then adding knowledge of the secondary criterion leads to replacing the acyclicity requirement with a test of consistency between revealed and observed second-stage superiority (Propositions 4.2 and 4.3). On the other hand, adding knowledge of the remaining first-stage variable requires both strengthening the acyclicity condition and checking compatibility of choices with the observed primary profile (Proposition 4.6).

5 Discussion

5.1 Summary of results

Our axiomatizations of the TST model with partial knowledge are summarized in Figure 1 for multi-valued choice functions and in Figure 2 under the single-valuedness restriction. These figures also show the logical relationships among the various results. For example, starting from Theorem 3.18 (resp., Proposition 4.2), we can discard our knowledge of the secondary criterion and thereby arrive at Theorem 3.5 (resp., Proposition 4.4). Hence any choice data permitted by the former result must be permitted by the latter, as well as by Theorem 3.12 (resp., Proposition 4.1) since we can equally well discard our knowledge of the primary criterion. Moreover, we have already seen in Section 4 how each single-valued characterization is a corollary of the corresponding multi-valued result.

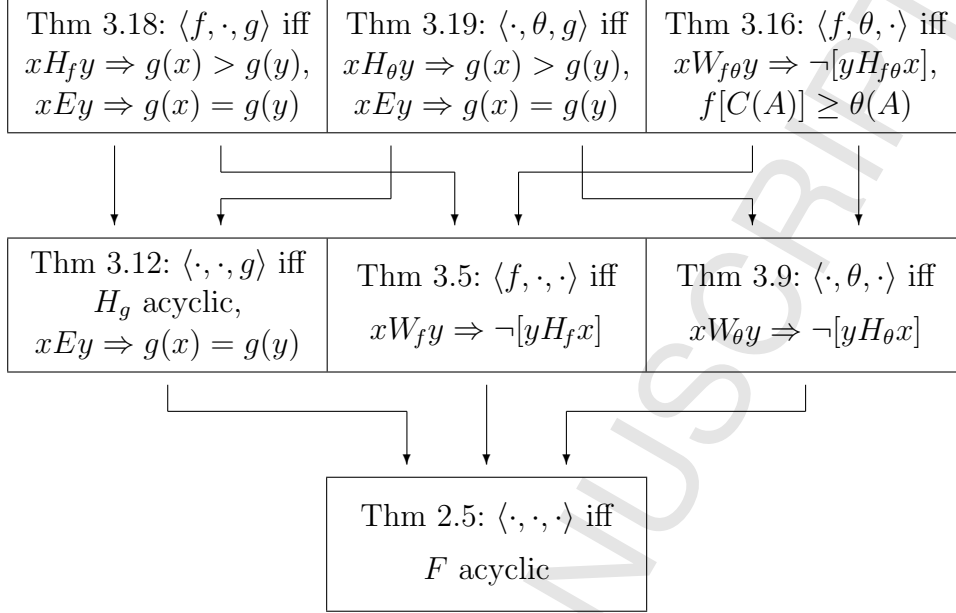


Figure 1: Summary of characterization results for multi-valued choice.

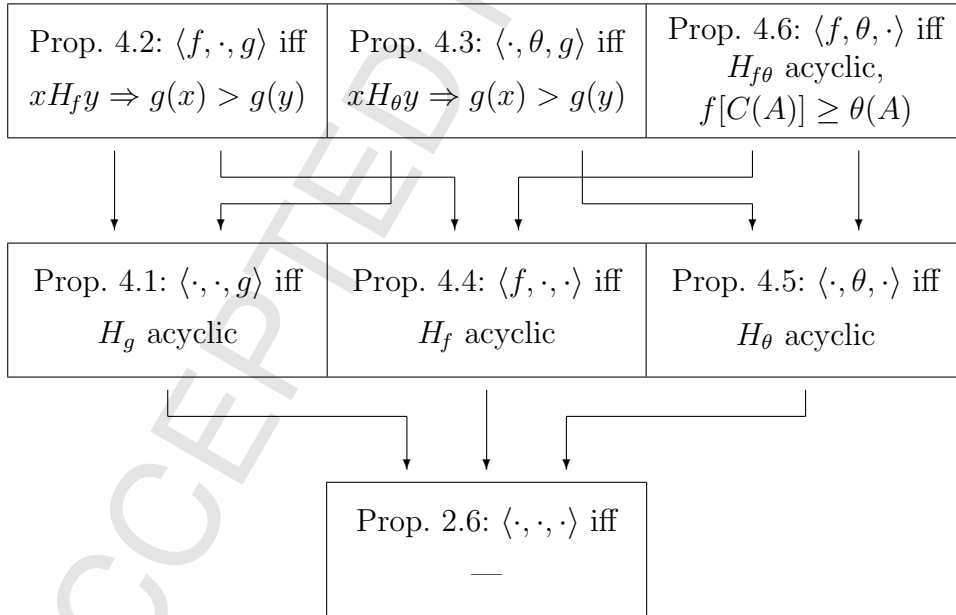


Figure 2: Summary of characterization results for single-valued choice.

5.2 Partial knowledge in other settings

In addition to establishing the specific results shown in Figures 1–2, a further goal of this paper has been to introduce the issue of partial knowledge itself. Beyond the TST context, partial-knowledge characterizations can be sought for other models of choice that make use of multiple “structural variables” in the sense of distinguishable cognitive components. For models that have features in common with the present framework—such as threshold effects or sequentially applied criteria—it may be hoped that our techniques will be transferable to some degree.

For instance, consider the following variant of the “rational shortlist method” (RSM) model proposed by Manzini and Mariotti [16]. In the first of two stages, the decision maker eliminates any alternative that is not maximal with respect to an asymmetric binary relation \succ .¹⁴ Then, in the second stage, a criterion function g is optimized in the usual way. Since maximization over menu A of an asymmetric \succ cannot in general be represented with a threshold structure $\Gamma(A|f, \theta)$, this model is not covered by the TST framework. Moreover, since optimization of a secondary criterion is stronger than the second-stage procedure specified by Manzini and Mariotti, the new model is a special case of an RSM.

This model would impose constraints on the decision maker’s behavior under either single- or multi-valued choice.¹⁵ Furthermore, we might wonder what *additional* restrictions are implied by knowledge of either the relation \succ or the criterion g . Assuming a known \succ would lead to a situation similar to that in Theorem 3.16, whose proof can be suitably modified (see Footnote 13). Alternatively, assuming a known g would lead to a situation resembling that in Theorem 3.12, and one that poses more of a challenge. Here the objective would be to use the known g and the observed C to infer information about the unknown \succ , devising axioms that rule out all possible contradictions.¹⁶

Partial knowledge could also be introduced in the setting of Salant and Rubinstein’s [26] “salient consideration functions.” Here choice sets have the structure $C(A) = \bigcup_{i=1}^n \{x \in A : \forall y \in A \neg[yP_i x]\}$, where n is an integer and each P_i a relation. Apart from the constraints intrinsic to this model, we can ask what additional restrictions are implied by knowledge of n , or of one or more P_i relations. And similar questions can be posed in the context of Kalai et al.’s [12] “rationalization by multiple rationales,” another prominent multiple-factor model of choice.

¹⁴Recall that a relation R is asymmetric if $xRy \Rightarrow \neg[yRx]$.

¹⁵Indeed, such constraints are implied by the general characterization of RSMs in [16].

¹⁶As a first step towards such an axiomatization, observe that if $x \in C(A)$, $y \in A \setminus C(A)$, and $g(y) \geq g(x)$, then for any $B \supseteq A$ we cannot have $y \in C(B)$.

A Appendix

As depicted in Figures 1–2, results for TST representations with more known variables can be used to help prove results with less known variables. For example, to demonstrate that the conditions in Theorem 3.12 are sufficient for a representation consistent with $\langle \cdot, \cdot, g \rangle$, it is enough to define a primary criterion f such that the conditions in Theorem 3.18 hold.

We shall make extensive use of this proof strategy, and so to preserve logical continuity we shall prove our results non-consecutively. Specifically, we prove first Theorem 3.18, followed by Theorems 3.5 and 3.12. We then prove Theorem 3.16, followed by Theorem 3.9. And lastly we prove Theorem 3.19.

A few items of notation not employed in the main text will be used in the proofs: We write xR^*y if $\exists z_1, z_2, \dots, z_n \in X$ such that $x = z_1 R z_2 R \dots R z_n = y$ (thereby defining the transitive closure R^* of the relation R). Furthermore, we write $K(x)$ for the E -equivalence class of $x \in X$, and $\mathcal{K} = \{K(x) : x \in X\}$ for the associated partition of X .¹⁷

Proof of Theorem 3.18. Let $\langle f, \theta, g \rangle$ be a TST representation of C , whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. For $x, y \in X$, if $xH_f y$ then $f(y) \geq f(x)$ and xSy . Hence $\exists A \in \mathcal{D}$ with $x \in C(A)$ and $y \in A \setminus C(A)$, so that $f(y) \geq f(x) \geq \theta(A)$ and $g(x) > g(y)$. Thus $xH_f y \Rightarrow g(x) > g(y)$.

Conversely, suppose that both $xH_f y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$. Given $A \in \mathcal{D}$, let $\theta(A) = \min_{x \in C(A)} f(x)$, so that for each $x \in C(A)$ we have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$ we have xTy , xEy , and $g(x) = g(y)$. Now let $w \in C(A)$ be such that $f(w) = \theta(A)$. If $\exists z \in A \setminus C(A)$ with $f(z) \geq \theta(A)$, then both wSz and $f(z) \geq \theta(A) = f(w)$. But then $wH_f z$ and so $g(w) > g(z)$. It follows that $\langle f, \theta, g \rangle$ is a TST representation of C . \square

Proof of Theorem 3.5. Let $\langle f, \theta, g \rangle$ be a TST representation of C . We then have $xH_f y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$ by Theorem 3.18, and it follows that $xW_f y \Rightarrow g(x) \geq g(y) \Rightarrow \neg[yH_f x]$.

Conversely, suppose that $xW_f y \Rightarrow \neg[yH_f x]$. For $K_1, K_2 \in \mathcal{K}$, let $K_1 \gg K_2$ if there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1 H_f x_2$.

A.1 Lemma. \gg is acyclic.

Proof. Suppose instead that $\exists K_1, K_2, \dots, K_n \in \mathcal{K}$ with $K_1 \gg K_2 \gg \dots \gg K_n \gg K_1$. For each $k \in \{1, 2, \dots, n\}$ there must exist $x_k, y_k \in K_k$ such that

¹⁷Recall that a binary relation is an equivalence if it is reflexive, symmetric, and transitive. (Reflexivity of R means that xRx , and symmetry means that $xRy \Rightarrow yRx$.) Extended togetherness inherits the properties of reflexivity and symmetry from togetherness, and is transitive by construction.

$x_1 H_f y_2 E x_2 H_f y_3 E \cdots H_f y_n E x_n H_f y_1 E x_1$. But then both $y_2 W_f x_1$ and $x_1 H_f y_2$, contradicting $y_2 W_f x_1 \Rightarrow \neg[x_1 H_f y_2]$. \square

Since \gg is acyclic, \gg^* is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a linear order \ggg such that $\forall K_1, K_2 \in \mathcal{K}$ we have $K_1 \gg K_2 \Rightarrow K_1 \ggg K_2$. Now let xQy if $K(x) \ggg K(y)$, so that Q is a weak order, and take any numerical representation g of Q .¹⁸

For $x, y \in X$ we have xEy only if $K(x) = K(y)$ and $g(x) = g(y)$. Moreover, $xH_f y$ only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and $g(x) > g(y)$. But then C has a TST representation consistent with $\langle f, \cdot, g \rangle$ by Theorem 3.18. \square

Proof of Theorem 3.12. Let $\langle f, \theta, g \rangle$ be a TST representation of C , whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. Moreover, for $x, y \in X$ we have $xH_f y \Rightarrow g(x) > g(y)$ by Theorem 3.18, which is logically equivalent to $xH_g y \Rightarrow f(x) > f(y)$. But then H_g is acyclic.

Conversely, suppose that both H_g is acyclic and $xEy \Rightarrow g(x) = g(y)$. Let xQy if $g(y) > g(x)$ or $xH_g y$, so that $\forall w, z \in X$ we have $wQ^*z \Rightarrow g(z) \geq g(w)$.

A.2 Lemma. Q is acyclic.

Proof. Suppose instead that $\exists x_1, x_2, \dots, x_n \in X$ such that $x_1 Q x_2 Q \cdots Q x_n = x_1$. Since H_g is acyclic, there must exist a $k < n$ such that $g(x_{k+1}) > g(x_k)$. But since $x_{k+1} Q^* x_k$ we have also $g(x_k) \geq g(x_{k+1})$, a contradiction. \square

Since Q is acyclic, Q^* is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a weak order P such that $\forall x, y \in X$ we have $xQ^*y \Rightarrow xPy$. Let f be any numerical representation of P .

For $x, y \in X$ we now have $xH_f y$ only if xSy and $f(y) \geq f(x)$; and thus only if $\neg[xPy]$, $\neg[xQ^*y]$, $\neg[xQy]$, and $g(x) > g(y)$. But then C has a TST representation consistent with $\langle f, \cdot, g \rangle$ by Theorem 3.18. \square

Proof of Theorem 3.16. Let $\langle f, \theta, g \rangle$ be a TST representation of C , whereupon the implications $x \in C(A) \Rightarrow f(x) \geq \theta(A)$ and $xEy \Rightarrow g(x) = g(y)$ are both immediate. For $x, y \in X$, if $xH_{f\theta} y$ then $\exists A \in \mathcal{D}$ such that $f(y) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$, which implies $g(x) > g(y)$. And it follows that $xW_{f\theta} y \Rightarrow g(x) \geq g(y) \Rightarrow \neg[yH_{f\theta} x]$.

Conversely, suppose that both $xW_{f\theta} y \Rightarrow \neg[yH_{f\theta} x]$ and $f[C(A)] \geq \theta(A)$. For $K_1, K_2 \in \mathcal{K}$, let $K_1 \gg K_2$ if there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $x_1 H_{f\theta} x_2$.

¹⁸Recall that a binary relation is a strict partial order if it is irreflexive and transitive; a weak order if it is a strict partial order that is negatively transitive; and a linear order if it is a weak order that is weakly complete. (Irreflexivity of R means that $\neg[xRx]$; negative transitivity means that $\neg[xRy] \wedge \neg[yRz] \Rightarrow \neg[xRz]$; and weak completeness means that $\neg[xRy] \wedge \neg[yRx] \Rightarrow x = y$.)

A.3 Lemma. \gg is acyclic.

Proof. Suppose instead that $\exists K_1, K_2, \dots, K_n \in \mathcal{K}$ with $K_1 \gg K_2 \gg \dots \gg K_n \gg K_1$. For each $k \in \{1, 2, \dots, n\}$ there must exist $x_k, y_k \in K_k$ such that $x_1 H_{f\theta} y_2 E x_2 H_{f\theta} y_3 E \dots H_{f\theta} y_n E x_n H_{f\theta} y_1 E x_1$. But then both $y_2 W_{f\theta} x_1$ and $x_1 H_{f\theta} y_2$, contradicting $y_2 W_{f\theta} x_1 \Rightarrow \neg[x_1 H_{f\theta} y_2]$. \square

Since \gg is acyclic, \gg^* is a strict partial order. By Szpilrajn's Theorem [29], it follows that there exists a linear order \ggg such that $\forall K_1, K_2 \in \mathcal{K}$ we have $K_1 \gg K_2 \Rightarrow K_1 \ggg K_2$. Now let xQy if $K(x) \ggg K(y)$, so that Q is a weak order, and take any numerical representation g of Q .

For $x, y \in X$ we have xEy only if $K(x) = K(y)$ and $g(x) = g(y)$. Moreover, we have $xH_{f\theta}y$ only if $K(x) \gg K(y)$, $K(x) \ggg K(y)$, and $g(x) > g(y)$.

Given $A \in \mathcal{D}$ and $x \in C(A)$, we have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$, we have xTy , xEy , and $g(x) = g(y)$. If there exists a $z \in A \setminus C(A)$ with $f(z) \geq \theta(A)$, then we have $xH_{f\theta}z$ and so $g(x) > g(z)$. It follows that $\langle f, \theta, g \rangle$ is a TST representation of C . \square

Proof of Theorem 3.9. Let $\langle f, \theta, g \rangle$ be a TST representation of C . We then have $xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x]$ and $f[C(A)] \geq \theta(A)$ by Theorem 3.16. It follows that $\forall x \in X$ we have $f(x) \geq M(x|\theta)$. Moreover, for $x, y \in X$ we have $xH_{\theta}y$ only if $\exists A \in \mathcal{D}$ such that $f(y) \geq M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$, which implies $xH_{f\theta}y$. Hence $xW_{\theta}y \Rightarrow xW_{f\theta}y \Rightarrow \neg[yH_{f\theta}x] \Rightarrow \neg[yH_{\theta}x]$.

Conversely, suppose that $xW_{\theta}y \Rightarrow \neg[yH_{\theta}x]$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $f(x) \geq \theta(A)$. Moreover, for each $x, y \in X$ we have $xH_{\theta}y \iff xH_{f\theta}y$ and hence $xW_{\theta}y \iff xW_{f\theta}y$. But then we can conclude that $xW_{f\theta}y \Rightarrow xW_{\theta}y \Rightarrow \neg[yH_{\theta}x] \Rightarrow \neg[yH_{f\theta}x]$, and so C has a TST representation consistent with $\langle f, \theta, \cdot \rangle$ by Theorem 3.16. \square

Proof of Theorem 3.19. Let $\langle f, \theta, g \rangle$ be a TST representation of C , whereupon the implication $xEy \Rightarrow g(x) = g(y)$ is immediate. Moreover, for $x, y \in X$ we have $xH_{\theta}y$ only if $\exists A \in \mathcal{D}$ with $M(y|\theta) \geq \theta(A)$, $x \in C(A)$, and $y \in A \setminus C(A)$. Now let $B \in \mathcal{D}$ be such that $y \in C(B)$ and $M(y|\theta) = \theta(B)$. It follows that $f(y) \geq \theta(B) = M(y|\theta) \geq \theta(A)$, and so $g(x) > g(y)$. Thus $xH_{\theta}y \Rightarrow g(x) > g(y)$.

Conversely, suppose that both $xH_{\theta}y \Rightarrow g(x) > g(y)$ and $xEy \Rightarrow g(x) = g(y)$. For each $x \in X$, let $f(x) = M(x|\theta)$. Given $A \in \mathcal{D}$ and $x \in C(A)$, we then have $f(x) \geq \theta(A)$. Moreover, for any $y \in C(A)$ we have xTy , xEy , and $g(x) = g(y)$. If $\exists z \in A \setminus C(A)$ with $\theta(A) \leq f(z) = M(z|\theta)$, then $xH_{\theta}z$ and so $g(x) > g(z)$. It follows that $\langle f, \theta, g \rangle$ is a TST representation of C . \square

References

- [1] Attila Ambrus and Kareen Rozen. Rationalising choice with multi-self models. *The Economic Journal*, 125(585):1136–1156, June 2015.
- [2] Robert J. Aumann. Utility theory without the completeness axiom. *Econometrica*, 30(3):445–462, July 1962.
- [3] Nick Baigent and Wulf Gaertner. Never choose the uniquely largest: A characterization. *Economic Theory*, 8(2):239–249, August 1996.
- [4] Gent Bajraj and Levent Ülkü. Choosing two finalists *and* the winner. *Social Choice and Welfare*, 45(4):729–744, December 2015.
- [5] Truman F. Bewley. Knightian decision theory: Part I. *Decisions in Economics and Finance*, 25(2):79–110, November 2002.
- [6] Walter Bossert and Yves Sprumont. Non-deteriorating choice. *Economica*, 76(302):337–363, April 2009.
- [7] Andrew Caplin and Mark Dean. Search, choice, and revealed preference. *Theoretical Economics*, 6(1):19–48, January 2011.
- [8] Vadim Cherepanov, Timothy Feddersen, and Alvaro Sandroni. Rationalization. *Theoretical Economics*, 8(3):775–800, September 2013.
- [9] Kfir Eliaz and Efe A. Ok. Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences. *Games and Economic Behavior*, 56(1):61–86, July 2006.
- [10] Kfir Eliaz, Michael Richter, and Ariel Rubinstein. Choosing the two finalists. *Economic Theory*, 46(2):211–219, February 2011.
- [11] Kfir Eliaz and Ran Spiegler. Consideration sets and competitive marketing. *Review of Economic Studies*, 78(1):235–262, January 2011.
- [12] Gil Kalai, Ariel Rubinstein, and Ran Spiegler. Rationalizing choice functions by multiple rationales. *Econometrica*, 70(6):2481–2488, November 2002.
- [13] J. Sebastian Lleras, Yusufcan Masatlioglu, Daisuke Nakajima, and Erkut Y. Ozbay. When more is less: Limited consideration. Unpublished, October 2010.

- [14] Michael Mandler. Indifference and incompleteness distinguished by rational trade. *Games and Economic Behavior*, 67(1):300–314, September 2009.
- [15] Michael Mandler, Paola Manzini, and Marco Mariotti. A million answers to twenty questions: Choosing by checklist. *Journal of Economic Theory*, 147(1):71–92, January 2012.
- [16] Paola Manzini and Marco Mariotti. Sequentially rationalizable choice. *American Economic Review*, 97(5):1824–1839, December 2007.
- [17] Paola Manzini and Marco Mariotti. Choice by lexicographic semiorders. *Theoretical Economics*, 7(1):1–23, January 2012.
- [18] Paola Manzini, Marco Mariotti, and Christopher J. Tyson. Two-stage threshold representations. *Theoretical Economics*, 8(3):875–882, September 2013.
- [19] Marco Mariotti. What kind of preference maximization does the weak axiom of revealed preference characterize? *Economic Theory*, 35(2):403–406, May 2008.
- [20] Yusufcan Masatlioglu and Daisuke Nakajima. Choice by iterative search. *Theoretical Economics*, 8(3):701–728, September 2013.
- [21] Yusufcan Masatlioglu, Daisuke Nakajima, and Erkut Y. Ozbay. Revealed attention. *American Economic Review*, 102(5):2183–2205, August 2012.
- [22] Yusufcan Masatlioglu and Efe A. Ok. Rational choice with status quo bias. *Journal of Economic Theory*, 121(1):1–29, March 2005.
- [23] Hiroki Nishimura and Efe A. Ok. Utility representation of an incomplete and nontransitive preference relation. Unpublished, February 2015.
- [24] Marcel K. Richter. Revealed preference theory. *Econometrica*, 34(3):635–645, July 1966.
- [25] Ariel Rubinstein and Yuval Salant. A model of choice from lists. *Theoretical Economics*, 1(1):3–17, March 2006.
- [26] Yuval Salant and Ariel Rubinstein. (A, f) : Choice with frames. *Review of Economic Studies*, 75(4):1287–1296, October 2008.

- [27] Herbert A. Simon. A behavioral model of rational choice. *Quarterly Journal of Economics*, 69(1):99–118, February 1955.
- [28] Dean Spears. Intertemporal bounded rationality as consideration sets with contraction consistency. *The B.E. Journal of Theoretical Economics: Contributions*, 11(1), June 2011.
- [29] Edward Szpilrajn. Sur l’extension de l’ordre partiel. *Fundamenta Mathematica*, 16:386–389, 1930.
- [30] Christopher J. Tyson. Cognitive constraints, contraction consistency, and the satisficing criterion. *Journal of Economic Theory*, 138(1):51–70, January 2008.
- [31] Christopher J. Tyson. Satisficing behavior with a secondary criterion. *Social Choice and Welfare*, 44(3):639–661, March 2015.